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On the stability number of claw-free P_5-free and more general graphs

Andreas Brandstädt[a,*,1] Peter L. Hammer[b]

[a] *Universität Rostock Fachbereich Informatik, Albert-Einstein-Str.21, D 18051 Rostock Germany*
[b] *RUTCOR, Rutgers University, New Brunswick NJ, USA*

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Abstract

In this note we show that the stability number of a (4-pan, chair, $K_{1,4}, P_5$)-free graph which has no simplicial vertex is bounded by 3. This generalizes the case of (claw, P_5)-free graphs and leads to a very simple polynomial-time algorithm for determining the stability number of (claw, P_5)-free graphs and, more generally, of (4-pan,chair, $K_{1,4}$, P_5)-free graphs. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The *stability number* $\alpha(G)$ of an undirected simple graph G is the maximum number of pairwise nonadjacent vertices in G. The problem of computing the stability number for a given graph has attracted much interest, and it is well-known that this problem is *NP*-hard and even hard to approximate. Basic examples of graph classes for which this problem can be solved in polynomial time are the perfect graphs together with some special cases such as chordal graphs [2] and the claw-free graphs [5,7,4]. For the special case of claw- and net-free graphs a method called *struction* (stability number reduction) works well [3]. Another interesting case – the bull- and chair-free graphs – was treated in [1]. For the chordal graphs the notion of simplicial vertices plays a crucial role. A vertex v is *simplicial* if $N(v)$ is a clique. For each simplicial vertex

* Corresponding author. fax: +49 381 498 3366.

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E-mail addresses: ab@informatik.uni-rostock.de (A. Brandstädt), hammer@rutcor.rutgers.edu (P.L. Hammer)

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v obviously $\alpha(G)=\alpha(G\setminus N[v])+1$ holds. Thus, given a simplicial vertex, it is easy to reduce the problem of determining $\alpha(G)$ to the same problem on a smaller graph. This suggests the following trivial algorithm. Hereby assume that G is not chordal; otherwise, $\alpha(G)$ can be found in linear time [6]. The reason for the size bound four in the subsequent step (3) is purely technical and hopefully can be generalized to larger constants and larger classes of graphs for which Algorithm 1 correctly finds the stability number.

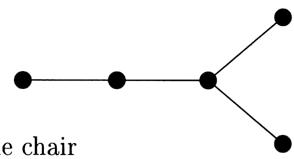
Algorithm 1. (1) $k{:=}0; G'{:=}G$;
(2) As long as there is a simplicial vertex v in G' set $k{:=}k+1$ and $G'{:=}G'\setminus N_{G'}[v]$;
(3) Check whether G' contains a stable set of four vertices. If not, then determine $\alpha(G')$ and set $\alpha(G)=k+\alpha(G')$.

Note that due to the assumption that G is not chordal we end up in step (3) with a nonempty graph G'. If now there is a stable set of size four we are in the general case which is *NP*-hard – we will show that in this case the graph G is not in the class of (4-pan, chair, $K_{1,4}, P_5$)-free graphs – a slight generalization of (claw, P_5)-free graphs. Otherwise the problem becomes simple – we will show that for (4-pan, chair, $K_{1,4}, P_5$)-free graphs the second case of step (3) applies: these graphs have either a simplicial vertex or stability number bounded by 3.

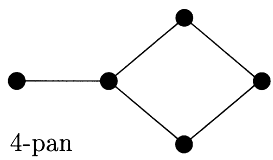
Note that Algorithm 1 is time-bounded by $\mathcal{O}(n^4)$ if G contains n vertices: For a given graph G it can be tested in time $\mathcal{O}(n^3)$ whether G contains a simplicial vertex. This has to be done at most n times until the remaining graph G' has no simplicial vertex. Then step (3) can be tested again in time $\mathcal{O}(n^4)$. If G' contains no stable set of four vertices then $\alpha(G')\leqslant 3$ and can be determined in time $\mathcal{O}(n^3)$.

Now to some further notions.

Vertices $a, b_1, \ldots, b_k$ form a star $K_{1,k}$ if $b_1, \ldots, b_k$ are pairwise nonadjacent and for all $i \in \{1, \ldots, k\}$ a is adjacent to b_i. The special case $k=3$ is the *claw* (or $K_{1,3}$). As usual, P_k denotes an induced path with k vertices and $k-1$ edges. A *chair* consists of five vertices a, b, c, d, e and edges ab, bc, cd, ce. A 4-*pan* consists of five vertices a, b, c, d, e and edges ab, bc, bd, ce, de. Obviously, claw-free graphs are (4-pan, chair, $K_{1,4}$)-free.



The chair



The 4-pan

The *distance* $d(u, v)$ between two vertices $u, v \in V$ is the length (number of edges) of a shortest path between u and v.

For vertex v let $N^i(v)=\{u\colon u\in V \text{ and } d(u,v)=i\}$. The *hanging* of G with start vertex v is the partition of V into nonempty levels $N(v)=N^1(v), N^2(v), \ldots$ (note that there are no edges between $N^i(v)$ and $N^j(v)$ for $|i-j|>1$).

2. Diameter 3

In a P_5-free graph for every vertex v and every $k \geqslant 4$ the kth level $N^k(v)$ of a hanging is empty. We first consider the case of vertices v with $N^3(v) \neq \emptyset$.

Lemma 1. *In a* (4-pan, chair, P_5)*-free graph, every vertex v with $N^3(v) \neq \emptyset$ is simplicial.*

Proof. Assume that v is not simplicial and let $a, b \in N(v)$ with $a, b \notin E$. Since $N^3(v) \neq \emptyset$ there is a neighbour $c \in N(v)$ which extends to a P_4 $vcde$ with $d \in N^2(v)$, $e \in N^3(v)$ (a *vertical* P_4). Assume first that $c = b$. Since G is P_5-free $avbde$ is no P_5. The only possible chord is ad but then the vertices a, v, c, d, e induce a 4-pan in G – a contradiction. Now assume that for no independent pair of neighbours a, b of v a or b extends into a vertical P_4. Let c be again a neighbour of v which extends into a vertical P_4 $vcde$. Then $ac \in E$, $bc \in E$ and $ad \notin E$, $bd \notin E$, but now the vertices a, b, c, d, e induce a chair – a contradiction. Thus v is simplicial. □

Note that this can be easily generalized to the following

Corollary 1. *In a* (4-pan, chair, P_{k+2})*-free graph*, $k \geqslant 3$, *every vertex v with $N^k(v) \neq \emptyset$ is simplicial.*

3. Diameter 2

Now assume that G is a (4-pan, chair, P_5)-free graph without simplicial vertices. Then, due to Lemma 1 all vertices have pairwise distance at most 2.

Lemma 2. *Let $G = (V, E)$ be a* (4-pan, chair, $K_{1,4}, P_5$)*-free graph with the property that no vertex is simplicial. Then $\alpha(G) \leqslant 3$ holds.*

Proof. Assume that G has a stable set $\{v, a, b, c\}$ of size 4. Let v be the start vertex of a hanging. Thus $a, b, c \in N^2(v)$. Note that a, b, c cannot have a common neighbour in $N(v)$.

Case 1: Every pair y, z of vertices from a, b, c has a common neighbour x_{yz} in $N(v)$. These x-vertices are pairwise adjacent, otherwise G would contain a P_5. Since v is not simplicial, it has two nonadjacent neighbours.

Case 1.1: There is a vertex $u \in N(v)$ which is nonadjacent to one of the x-vertices. Assume w.l.o.g. that $ux_{bc} \notin E$. If $ub \notin E$ and $uc \notin E$ then u, v, x_{bc}, b, c induce a chair. If $ub \notin E$ and $uc \in E$ then u, v, x_{bc}, b, c induce a 4-pan, and the same for $ub \in E$ and $uc \notin E$. Thus $ub \in E$ and $uc \in E$ holds. Note that $ua \notin E$ holds. If $ux_{ac} \notin E$ then u, v, c, x_{ac}, a induce a 4-pan, if $ux_{ab} \notin E$ then u, v, b, x_{ab}, a induce a 4-pan. Thus $ux_{ac} \in E$ and $ux_{ab} \in E$. But now u, b, x_{bc}, x_{ac}, a induce a 4-pan – a contradiction.

Case 1.2: There are nonadjacent vertices $u, u' \in N(v)$ which are adjacent to all of the x-vertices. Note that neither u nor u' is adjacent to all of the vertices v, a, b, c.

None of the vertex sets $\{u, u', x_{ab}, a, b\}$, $\{u, u', x_{ac}, a, c\}$, $\{u, u', x_{bc}, b, c\}$ induces a $K_{1,4}$. Thus there are further edges incident to u, u'.

Claim. *u and u' do not have a common neighbour in a, b, c.*

Proof of the claim. Assume that a is a common neighbour of u and u'. Since u, a, x_{bc}, u', c is no 4-pan, $uc \in E$ or $u'c \in E$ holds. Since u, a, x_{bc}, u', b is no 4-pan, $ub \in E$ or $u'b \in E$ holds. Note that neither $ub, uc \in E$ nor $u'b, u'c \in E$.

Assume that $uc, u'b \in E$ (and thus $ub, u'c \notin E$). Then $bu'auc$ is a P_5 – a contradiction. Analogously, in the case $u'c, ub \in E$ $cu'aub$ is a P_5 – a contradiction. □

Assume now that $ua \in E$ and thus due to the claim $u'a \notin E$. Since a, u, x_{bc}, u', b induce no chair, either $ub \in E$ or $u'b \in E$. Since a, u, x_{bc}, u', c induce no chair, either $uc \in E$ or $u'c \in E$. Since a, u, x_{bc}, c, b induce no chair, either $ub \in E$ or $uc \in E$. If $ub \notin E$ then $uc \in E$, $u'b \in E$ and $u'c \notin E$, $u'a \notin E$. But then $bu'vua$ is a P_5. If $ub \in E$ then $uc \notin E$, $u'b \notin E$ and $u'c \in E$. But then $cu'vub$ is a P_5 – a contradiction.

Case 2: There is a pair y, z of vertices from a, b, c without common neighbour in $N(v)$. Assume w.l.o.g. that a and b have no common neighbour in $N(v)$. Then there is a common neighbour $d \in N^2(v)$ of a and b. Let a' (b') be a neighbour of a (b) in $N(v)$. Since $aa'vb'b$ is no P_5 the vertices a' and b' are adjacent. Moreover, since $va'adb$ and $vb'bda$ are no P_5, $a'd \in E$ and $b'd \in E$.

Case 2.1: Let $cd \in E$. Since v, b', d, a, c induce no chair, $b'c \in E$. Since v, a', d, b, c induce no chair, $a'c \in E$. Now, changing the role of v and c and recalling that c is not simplicial, we are in Case 1 for a, b, v instead of a, b, c.

Case 2.2: Let $cd \notin E$. If $ca' \in E$ then b, d, a', c, v induces a chair, if $cb' \in E$ then a, d, b', c, v induces a chair. Thus $ca' \notin E$ and $cb' \notin E$. Let $c' \in N(v)$ be adjacent to c. Since $bb'vc'c$ is no P_5, $b'c' \in E$ or $bc' \in E$, but if $b'c' \notin E$ and $bc' \in E$, then b, b', v, c', c would induce a 4-pan. Thus $b'c' \in E$. Since $cc'b'da$ is no P_5, $ac' \in E$ or $dc' \in E$, but if $dc' \notin E$ and $ac' \in E$, then a, d, b', c', c would induce a 4-pan. Thus $dc' \in E$. Now for $ac' \notin E$ a, d, c', v, c would induce a chair, thus $ac' \in E$ and consequently $bc' \notin E$. But then b, b', c', a, c induce a chair – a contradiction. □

4. Final remarks

Lemmas 1 and 2 guarantee that Algorithm 1 determines the stability number of (4-pan, chair, $K_{1,4}, P_5$)-free (and thus of claw-free P_5-free) graphs. Note that it is not necessary to check in advance whether the input graph G is (4-pan, chair, $K_{1,4}, P_5$)-free: if Algorithm 1 finds in step (3) a stable set of size four then G is not (4-pan, chair, $K_{1,4}, P_5$)-free, otherwise Algorithm 1 correctly finds $\alpha(G)$.

Note that (claw, P_5)-free graphs are in general not perfect since the induced chordless cycle C_5 of 5 vertices is contained in this class.

Another observation is that in (claw, P_5)-free graphs for every vertex $v \in V$ $N^2(v)$ induces a P_4-free subgraph (i.e. a cograph). This can easily be generalized to the following

Lemma 3. *If $G=(V,E)$ is claw-free P_{k+2}-free then for $j \geqslant k-1$ $N^j(v)$ is P_4-free.*

Proof. Assume not. Let $abcd$ be a P_4 in $N^j(v)$, $j \geqslant k-1$, and $P=(x_0,\dots,x_{k-1})$ be a path with $v=x_0$, $a=x_{k-1}$ and $x_i \in N^i(v)$, $i=1,2,\dots,k$. Since G contains no P_{k+2} the path $x_0x_1\dots bc$ is no P_{k+2}, thus $x_{k-2}b$ (the only possible chord) is an edge. Also $x_0\dots x_{k-2}bcd$ is no P_{k+2}, thus it has again a chord which is a contradiction to the claw-freeness of G. □

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